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REGULAR POLYTOPES OF TYPE {4,4,3} AND {4,4,4}

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Abstract regular polytopes generalize the classical concept of a regular polytope and regular tessellation to more complicated combinatorial structures with a distinctive geometrical and topological flavour. In this paper the authors give an almost complete classification of the (universal) locally toroidal regular 4-polytopes of Schläfli types $\{4,4,3\}$ and $\{4,4,4\}$.

1. Introduction

In recent years the classical theory of regular polytopes (cf. Coxeter [7]) has been generalized in various directions. One such generalization is the concept of regular incidence-polytopes, which provides a suitable setting for combinatorial structures resembling the classical regular polytopes (cf. Danzer-Schulte [10]); see also McMullen [13], Grünbaum [12], Dress [11], Tits [22] and Buekenhout [2].

In [12] Grünbaum suggested studying abstract regular polytopes whose faces and vertex-figures are not necessarily of spherical type. He gave several examples and posed the problem of classifying the finite universal (or, in his terminology, naturally generated) abstract regular 4-polytopes $\{\mathcal{P}_1,\mathcal{P}_2\}$ with spherical and/or toroidal 3-faces $\mathcal{P}_1$ and vertex-figures $\mathcal{P}_2$; see also Coxeter-Shephard [9], Weiss [23], [24], and [18], [19]. In [16], [17] this "amalgamation" problem for $\mathcal{P}_1$ and $\mathcal{P}_2$ was attacked by means of so-called twisting operations on Coxeter groups and unitary reflexion groups. This technique proved to be the right tool to obtain a complete classification for many interesting classes of $\mathcal{P}_1$ and $\mathcal{P}_2$.

In this paper we study the classification problem for the abstract polytopes $\{\{4,4\}_{s,t},\{4,3\}\}$ and $\{\{4,4\}_{q,r},\{4,4\}_{s,t}\}$. We give a complete classification in the former case and an almost complete classification in the latter case. The only case we have not been able to settle is $\{\{4,4\}_{s,0},\{4,4\}_{t,0}\}$ with s and t distinct odd integers (≥ 3). Excluding possibly existing finite polytopes of this exceptional type, up to duality the only finite instances are those of Table 1.

In Section 5 we associate with each of the polytopes $\{\{4,4\}_{s,t},\{4,3\}\}$ and $\{\{4,4\}_{q,r},\{4,4\}_{s,t}\}$ a regular tessellation $\mathcal{H}$ on the euclidean 2-sphere or in the euclidean or hyperbolic plane, which in a sense cuts right through the polytope. Ex-

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cluding the exceptional case, the polytopes turn out to be finite if and only if $\mathcal{H}$ is spherical. This gives a simple geometric explanation why most polytopes are infinite.

2. Incidence-polytopes

Following [10], a *d-incidence-polytope* $\mathcal{P}$ is a partially ordered set, with a strictly monotone rank function dim(.) with range $\{-1, 0, \ldots, d\}$. The elements of rank i are called the *i-faces* of $\mathcal{P}$. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ all contain exactly $d+2$ faces, including the unique (least) (-1)-face F_{-1} and the unique (greatest) d-face F_d of $\mathcal{P}$. Further defining properties for $\mathcal{P}$ are the strong flag-connectedness as well as the homogeneity property that for any $(i-1)$-face F and any $(i+1)$-face G with $F < G$ there are exactly two i-faces H of $\mathcal{P}$ with $F < H < G$.

If F and G are faces with $F < G$, we call $G/F := \{H | F \le H \le G\}$ a *section* of $\mathcal{P}$. There is little possibility of confusion if we identify a face F with the section F/F_{-1}. The faces of dimension $0, 1$ or $d-1$ are also called *vertices, edges* and *facets* of $\mathcal{P}$, respectively. For a vertex F the section F_d/F is said to be the *vertex-figure* of $\mathcal{P}$ at F.

A d-incidence-polytope $\mathcal{P}$ is *regular* if its automorphism group $A(\mathcal{P})$ is transitive on the flags. For a regular $\mathcal{P}$, its group $A(\mathcal{P})$ is generated by involutions $\varrho_0, \ldots, \varrho_{d-1}$, where ϱ_i is the unique automorphism which keeps all but the i-face of some base flag of $\mathcal{P}$ fixed $(i = 0, \ldots, d-1)$. These *distinguished* generators of $A(\mathcal{P})$ satisfy relations

$$(1) \qquad (\varrho_i \varrho_j)^{p_{ij}} = 1 \; (i, j = 0, \ldots, d-1),$$

where $p_{ii} = 1$, $p_{ij} = p_{ji} =: p_{i+1}$ if $j = i+1$, and $p_{ij} = 2$ otherwise; here, the p_i's are given by the *(Schläfli-)type* of $\mathcal{P}$. The generators also have the *intersection property*:

$$(2) \qquad \langle \varrho_i | i \in I \rangle \cap \langle \varrho_i | i \in J \rangle = \langle \varrho_i | i \in I \cap J \rangle \quad \text{if } I, J \subset \{0, \ldots, d-1\}.$$

We call a group A with involutory generators $\varrho_0, \ldots, \varrho_{d-1}$ a *C-group* (C here stands for Coxeter) if it has properties (1) and (2). The C-groups are precisely the groups of regular incidence-polytopes (cf. [18]).

The regular 3-incidence-polytopes are (except for degenerate cases) precisely the (reflexible) regular maps on surfaces (cf. [8]). The only spherical regular maps are given by the Platonic solids $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$ and $\{5,3\}$. On the torus there are the three infinite series $\{4,4\}_{s,t}$, $\{3,6\}_{s,t}$ and $\{6,3\}_{s,t}$, with $s = t \ge 1$ or $t = 0, s \ge 2$; the maps $\{3,6\}_{1,1}$ and $\{6,3\}_{1,1}$ are incidence-polytopes, but $\{4,4\}_{1,1}$ is not.

Recall that a *Petrie-polygon* of a regular map $\mathcal{P}$ is a zig-zag along the edges such that any 2 but no 3 consecutive edges lie in a 2-face (cf. [8]). A *2-chain* of $\mathcal{P}$ is a path along edges which leaves, at each vertex, 2 2-faces to the right (cf. [4]). The lengths of the Petrie-polygons and 2-chains of $\mathcal{P}$ are the orders of the elements $\varrho_0\varrho_1\varrho_2$ and $\varrho_0\varrho_1\varrho_2\varrho_1$ in $A(\mathcal{P})$, respectively.

In this paper some incidence-polytopes are constructed by so-called *twisting operations* on Coxeter groups $W = \langle \sigma_0, \ldots, \sigma_m \rangle$ which admit suitable automorphism τ permuting the (canonical) generators σ_i. If the τ's are themselves involutions, we can augment W by their addition to construct a semi-direct product A of W by the group B generated by the τ's. In suitable cases A will be the group of a regular d-incidence-polytope; then the intersection property (2) follows from the corresponding property for W (cf. [1], Ch. 4, Section 1). A twisting operation $\varkappa$ on W will usually be denoted by

$$\varkappa : (\sigma_0, \dots, \sigma_m; \tau_0, \dots, \tau_k) \to (\varrho_0, \dots, \varrho_{d-1}),$$

where $\tau_0, \dots, \tau_k$ are the involutory generators of B and $\varrho_0, \dots, \varrho_{d-1}$ the distinguished generators of the new group A.

We shall also use other types of (mixing) operations on groups W generated by involutions $\sigma_0, \dots, \sigma_m$. Then we derive a new group A from W by taking as generators $\varrho_0, \dots, \varrho_{d-1}$ for A certain suitably chosen products of the generators σ_i of W; then A is a subgroup of W. A typical example is the *halving operation* η which applies to the group $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ of a regular map of type $\{4, q\}$ with $q \geq 3$; here, η is given by

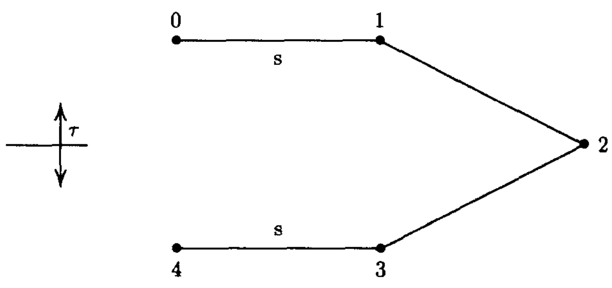
$$\eta : (\sigma_0, \sigma_1, \sigma_2) \to (\sigma_0\sigma_1\sigma_0, \sigma_2, \sigma_1) =: (\varrho_0, \varrho_1, \varrho_2)$$

(cf. [15], Section 4.3).

3. The type $\{4, 4, 3\}$

Following [12] we write $\mathcal{L}_{s,t} := \{\{4,4\}_{s,t}, \{4,3\}\}$, with $s = t \geq 2$ or $t = 0$, $s \geq$ ≥ 2. In [12] and [9] the universal $\mathcal{L}_{2,0}, \mathcal{L}_{3,0}$ and $\mathcal{L}_{2,2}$ were already recognized as finite incidence-polytopes; see also [3]. In [17], Section 6, we proved that $\mathcal{L}_{2,0}$ and $\mathcal{L}_{3,0}$ are in fact the only finite instances among the universal $\mathcal{L}_{s,0}$; for later use we briefly sketch the construction below. In this section we complete the classification for the type $\{4, 4, 3\}$ by showing that $\mathcal{L}_{2,2}$ is the only finite instance among the incidence-polytopes $\mathcal{L}_{s,s}$.

To construct $\mathcal{L}_{s,0}$ consider the Coxeter group $W = \langle \sigma_0, \dots, \sigma_4 \rangle$ with diagram

(3) 

(Here and below, if a mark $s = 2$ occurs, we regard the corresponding branch as missing). In (3) the symmetry of the diagram induces an automorphism τ of W permuting the σ_i. It was proved in [17] that the operation

(4) $$\varkappa : (\sigma_0, \dots, \sigma_4; \tau) \to (\sigma_0, \tau, \sigma_3, \sigma_2) =: (\varrho_0, \dots, \varrho_3)$$

gives the group $A = \langle \varrho_0, \dots, \varrho_3 \rangle$ of the universal $\mathcal{L}_{s,0}$. Hence $\mathcal{L}_{s,0}$ is finite if and only if W is finite, that is, $s = 2$ or $s = 3$. The groups of $\mathcal{L}_{2,0}$ and $\mathcal{L}_{3,0}$ are isomorphic to $D_4 \times S_4$ and $S_6 \times C_2$, respectively.

The following remark gives an explanation why $\mathcal{L}_{2,0}$ and $\mathcal{L}_{3,0}$ are finite but $\mathcal{L}_{s,0}$ with $s \geq 4$ is not. If we explicitly construct $\mathcal{L}_{s,0}$ by sticking together 3 tori $\{4,4\}_{s,0}$ around an edge and successively extending the arrangement by adding further tori, we observe that for $s \leq 3$ the arrangement closes up locally. For example, in constructing $\mathcal{L}_{s,0}$ we find that there is precisely one torus $\{4,4\}_{3,0}$ which surrounds the 3 tori like a belt. This does not occur if $s \geq 4$. See also Section 5.

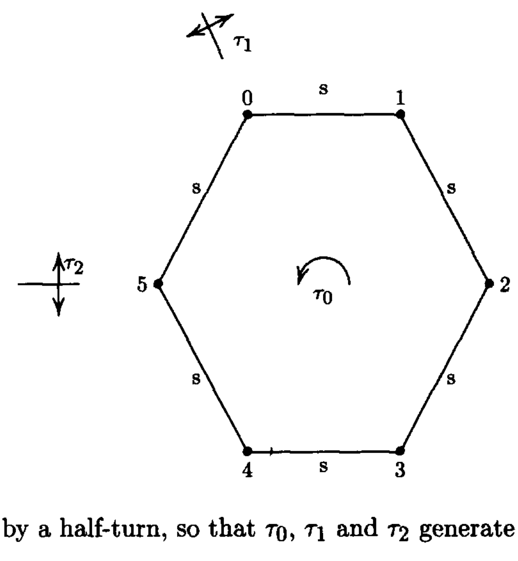
From $\mathcal{L}_{3,0}$ we can derive an easy example of non-existence of a universal $\{\mathcal{P}_1, \mathcal{P}_2\}$. We shall show that the universal $\mathcal{P} := \{\{4,4\}_{3,0}, \{4,3\}_3\}$ does not exist, that is, the group collapses. Here, $\{4,4\}_3$ denotes the 3-dimensional hemi-cube. In fact, if $\mathcal{P}$ exists, then its group is obtained from the group $A(\mathcal{L}_{3,0}) = \langle \varrho_0, \dots, \varrho_3 \rangle = S_6 \times C_2$ by introducing the extra relation

$$(\varrho_1 \varrho_2 \varrho_3)^3 = 1.$$

Since A_6 is the only non-trivial normal subgroup of S_6, this limits the choices of corresponding normal subgroups N of $A(\mathcal{L}_{3,0})$ to the second factor in $S_6 \times C_2$. In S_6 the automorphism $\tau = \varrho_1$ can be realized by inner conjugation with an involutory element σ(say); then in $A(\mathcal{L}_{3,0})$, $N = \langle \varrho_1 \sigma \rangle$. But $\varrho_1 \sigma \notin \langle \varrho_1, \varrho_2, \varrho_3 \rangle$, so that taking the quotient by N does not lead to a collapse of the vertex-figures $\{4,3\}$ of $\mathcal{L}_{3,0}$. In fact, $A(\mathcal{L}_{3,0})/N$ is the group of another regular incidence-polytope in the class $\langle \{4,4\}_{3,0}, \{4,3\} \rangle$, an "elliptic" quotient $\mathcal{L}_{3,0}/2$ of $\mathcal{L}_{3,0}$; this polytope was also discovered in [12].

For the construction of the universal $\mathcal{L}_{s,s} = \{\{4,4\}_{s,s}, \{4,3\}\}$ consider the Coxeter group $W = \langle \sigma_0, \dots, \sigma_5 \rangle$ defined by

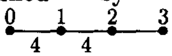
(5)



Here, τ_0 is induced by a half-turn, so that τ_0, τ_1 and τ_2 generate the dihedral group D_6. The operation

(6) $\qquad \varkappa : (\sigma_0, \dots, \sigma_5; \tau_0, \tau_1, \tau_2) \to (\tau_0, \sigma_2, \tau_1, \tau_2) = (\varrho_0, \dots, \varrho_3)$

gives us the group $A = \langle \varrho_0, \dots, \varrho_3 \rangle$ of a regular 4-incidence-polytope $\mathcal{P}$ of type $\{4,4,3\}$. Since $\langle \varrho_0, \varrho_1, \varrho_2 \rangle$ is a semi-direct product of $D_s \times D_s$ by $C_2 \times C_2$, the facets of

$\mathcal{P}$ are toroidal maps $\{4,4\}_{s,s}$. Computing a presentation we find that A is abstractly defined by the standard relations for the Coxeter group  and the extra relation

$$(\varrho_0\varrho_1\varrho_2)^{2s} = 1; \tag{7}$$

this extra relation follows from $\sigma_1\sigma_2 = (\varrho_2\varrho_0\varrho_1)^2$. Since (7) defines $\{4,4\}_{s,s}$, this proves that $\mathcal{P} = \mathcal{L}_{s,s}$.

The group of $\mathcal{L}_{s,s}$ is a semi-direct product of W by D_6, so that $\mathcal{L}_{s,s}$ is finite if and only if $s = 2$. In particular, $A(\mathcal{L}_{2,2})$ is the wreath product $C_2 \wr D_6$ (with D_6 acting as usual), of order 768. We summarize the results in a theorem.

Theorem 1. *The universal $\mathcal{L}_{s,t} := \{\{4,4\}_{s,t}, \{4,3\}\}$ exists for all pairs (s,t) with $s = t \geq 2$ or $t = 0$, $s \geq 2$. The only finite instances are $\mathcal{L}_{2,0}$, $\mathcal{L}_{3,0}$ and $\mathcal{L}_{2,2}$, with groups $D_4 \times S_4$, $S_6 \times C_2$ and $C_2 \wr D_6$, respectively.*

For $s = 3$ we can construct an infinite family of finite regular 4-incidence-polytopes $\mathcal{P}_1$ in the class $\langle\{4,4\}_{3,3}, \{4,3\}\rangle$ by applying the operation (6) to the finite unitary reflexion group $W = [1\ 1\ 4^p]^3$ represented by a hexagonal diagram (with a mark p inside to indicate one extra relation); see [5], [21] or [17]. Then, $A(\mathcal{P}_1)$ is a semi-direct product of $[1\ 1\ 4^p]^3$ by D_6, of order $8640p^5 (p \geq 2)$.

For $p \geq 3$ let $\mathcal{M}_p$ denote Coxeter's map $\{4, p|4^{[p/2]-1}\}$; then $\mathcal{M}_3 = \{4,3\}$ (cf. [4]). If $p \geq 5$ is odd and the operation (6) is generalized to $2p$-gonal diagrams (with $\varrho_1 = \sigma_2$ replaced by $\varrho_1 = \sigma_{p-1}$), then the resulting regular 4-incidence-polytopes belong to $\langle\{4,4\}_{2,2}, \mathcal{M}_p\rangle$. If $s = 2$, the group is $C_2 \wr D_{2p}$ (with $D_{2p} = \langle\tau_0, \tau_1, \tau_2\rangle$ acting as usual), of order $p2^{2p+1}$. In particular, $\{\{4,4\}_{2,2}, \mathcal{M}_p\}$ exists and is infinite. For the dual map $\mathcal{M}_p^*$ of $\mathcal{M}_p$ some existence and finiteness theorems on universal polytopes $\{\mathcal{N}, \mathcal{M}_p^*\}$ were discussed in [16], [17].

4. The type {4, 4, 4}

The discussion for the type $\{4,4,4\}$ involves the three cases $\{\{4,4\}_{s,0}, \{4,4\}_{t,0}\}$, $\{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$ and $\{\{4,4\}_{s,s}, \{4,4\}_{t,0}\}$, of which only the last is completely settled (see (d) below). For later use we recall some partial results.

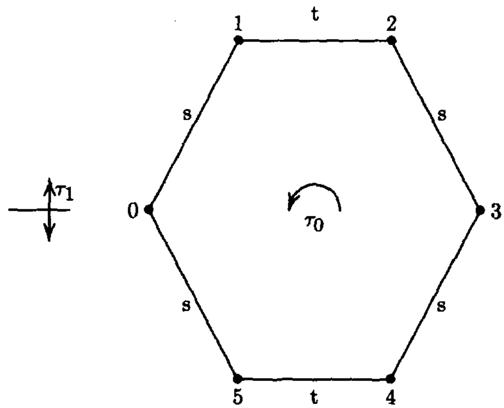
(a) By [3], the universal $\{\{4,4\}_{2,0}, \{4,4\}_{2s,0}\}$ and $\{\{4,4\}_{2,0}, \{4,4\}_{t,t}\}$ are known to exist and to be finite for all $s \geq 1$ and $t \geq 2$, with groups of order $128s^2$ and $64t^2$, respectively (see (d)).

(b) For $r \geq 2$ and $(s,t) \neq (2,0)$ the universal $\{\{4,4\}_{2r,0}, \{4,4\}_{s,t}\}$ exists, by [16], Theorems 5 and 6 and their corollaries. The only finite instances are $\{\{4,4\}_{4,0}, \{4,4\}_{s,0}\}$ for $s = 2$ or 3, with groups of order 512 and 36864, respectively.

(c) The universal $\{\{4,4\}_{2s,0}, \{4,4\}_{2t,0}\}$, $\{\{4,4\}_{2s,2s}, \{4,4\}_{2t,2t}\}$ and $\{\{4,4\}_{2s,2s}, \{4,4\}_{2t,0}\}$ exist for all $s,t \geq 1$, by [20], Sections 6 and 7.2.

(d) According to [17], Section 8.2, for $s, t \geq 2$ the universal $\{\{4,4\}_{s,0}, \{4,4\}_{t,t}\}$ can be constructed from the diagram

(8)



by the operation

$$\varkappa : (\sigma_0, \ldots, \sigma_5; \tau_0, \tau_1) \to (\sigma_3, \tau_0, \sigma_1, \tau_1) =: (\varrho_0, \ldots, \varrho_3). \tag{9}$$

Its group is a semi-direct product of the corresponding Coxeter group W by $C_2 \times C_2$. The only finite instances are $\{\{4,4\}_{3,0}, \{4,4\}_{2,2}\}$ and $\{\{4,4\}_{2,0}, \{4,4\}_{t,t}\}$ with $t \geq 2$, with groups of order 2304 and $64t^2$, respectively; see also [3]. Note that the group of $\{\{4,4\}_{2,0}, \{4,4\}_{t,t}\}$ is a quotient of $\{\{4,4\}_{2,0}, \{4,4\}_{2t,0}\}$; by (a), the corresponding normal subgroup has order 2 and thus consists of a central involution.

4.1. The case $\{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$

Let $\mathcal{P} := \{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$. We begin by discussing the case where both s and t are even.

Then, by (c), the existence of $\mathcal{P}$ is known. Also, by (d), we know the structure of $\mathcal{N} := \{\{4,4\}_{s,0}, \{4,4\}_{t,t}\}$. Clearly, $A(\mathcal{N})$ is a quotient of $A(\mathcal{P})$, so that $\mathcal{N}$ is obtained from $\mathcal{P}$ by identifications. In fact, $A(\mathcal{N})$ is derived from $A(\mathcal{P})$ by adding to the relations of $A(\mathcal{P})$ the extra relation

$$(\varrho_0\varrho_1\varrho_2\varrho_1)^s = 1 \tag{10}$$

which defines $\{4,4\}_{s,0}$. This proves that $\mathcal{P}$ is infinite if $\mathcal{N}$ is infinite. Hence, by (d), $\mathcal{P}$ is infinite if $s \geq 4$; note that s is even. By interchanging the roles of s and t we see that $\mathcal{P}$ can only be finite if $s = t = 2$. But in [3] the self-dual universal $\{\{4,4\}_{2,2}, \{4,4\}_{2,2}\}$ was already recognized as a finite incidence-polytope with group order 1024; for a description of the group see also Weiss [25] and [26]. This completes the discussion for even s and t.

Now, let at least one parameter, s (say), be odd. Again, let $\mathcal{N} := \{\{4,4\}_{s,0}, \{4,4\}_{t,t}\}$ and $A(\mathcal{N}) = \langle \tau_0, \ldots, \tau_3 \rangle$, with $\tau_0, \ldots, \tau_3$ the distinguished generators of $A(\mathcal{N})$.

Consider the direct product $A := A(\mathcal{N}) \times C_2$, with $C_2 = \langle \alpha \rangle$ (say); for $\tau \in$ $\in A(\mathcal{N})$ we write τ and $\tau\alpha (= \alpha\tau)$ instead of $(\tau, 1)$ and (τ, α), respectively. Then A

is generated by $\varrho_0 := \tau_0\alpha,\ \varrho_1 := \tau_1,\ \varrho_2 := \tau_2,\ \varrho_3 := \tau_3$. To see this, observe that by the structure of the facets of $\mathcal{N}$ and the choice of s we have

$$\alpha = (\tau_0\tau_1\tau_2\tau_1)^s\alpha^s = (\varrho_0\varrho_1\varrho_2\varrho_1)^s \in \langle\varrho_0, \varrho_1, \varrho_2\rangle. \tag{11}$$

From (11) we also deduce that $\langle\varrho_0, \varrho_1, \varrho_2\rangle$ is the group of the map $\{4,4\}_{s,s}$; in particular, $\langle\varrho_0, \varrho_1, \varrho_2\rangle = \langle\tau_0, \tau_1, \tau_2\rangle \times \langle\alpha\rangle$.

Our considerations imply that A is in fact a C-group; that is, A has property (2). The corresponding regular 4-incidence-polytope $\mathcal{L}$ belongs to $\langle\{4,4\}_{s,s}, \{4,4\}_{t,t}\rangle$. In particular this proves that the universal $\mathcal{P} = \{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$ exists. As before, $\mathcal{P}$ is infinite if $\mathcal{N}$ is infinite. Hence, since s is odd, (d) implies that $\mathcal{P}$ is infinite if $(s,t) \neq (3,2)$. But by [3] the universal $\{\{4,4\}_{3,3}, \{4,4\}_{2,2}\}$ is finite, with a group of order 9216. Note that $A = A(\mathcal{L})$ has index 2 in this group. We sum up the results in a theorem.

Theorem 2. *The universal incidence-polytopes $\{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$ exist for all $s,t \geq$ ≥ 2. Up to duality, the only finite instances are obtained for $(s,t) = (2,2)$ and $(3,2)$, with groups of order* 1024 *and* 9216, *respectively.*

4.2. The case $\{\{4,4\}_{s,0}, \{4,4\}_{t,0}\}$

Let $\mathcal{P} := \{\{4,4\}_{s,0}, \{4,4\}_{t,0}\}$. We being by discussing the case $s = 2$ and t odd, which is not covered by (a).

We shall see that for odd t the universal $\mathcal{P} = \{\{4,4\}_{2,0}, \{4,4\}_{t,0}\}$ does not exist because the group collapses. Assume to the contrary that $\mathcal{P}$ exists. Then $\mathcal{P}$ can be obtained from the universal $\mathcal{L} := \{\{4,4\}_{2,0}, \{4,4\}_{t,t}\}$ by making identifications; let $\varrho_0, \ldots, \varrho_3$ be the distinguished generators of $A(\mathcal{L})$. Since the identifications cause a collapse of the vertex-figure of $\mathcal{L}$ to $\{4,4\}_{t,0}$, the corresponding normal subgroup $\mathcal{N}$ of $A(\mathcal{L})$ must contain the element

$$\alpha := (\varrho_1\varrho_2\varrho_3\varrho_2)^t.$$

Let $t = 2m+1$, $m \geq 1$.

Now, using the construction (8) and (9) for $\mathcal{L}$ we see that

$$\alpha = (\tau_0\sigma_1\tau_1\sigma_1)^{2m+1} = (\sigma_4\sigma_5)^m\sigma_4(\sigma_2\sigma_1)^m\sigma_2\tau_0\tau_1.$$

Then, since $s = 2$, we have

$$\sigma_3\alpha\sigma_3 = (\sigma_4\sigma_5)^m\sigma_4(\sigma_2\sigma_1)^m\sigma_2(\sigma_3\tau_0\tau_1\sigma_3) = \alpha\sigma_0\sigma_3 \in N.$$

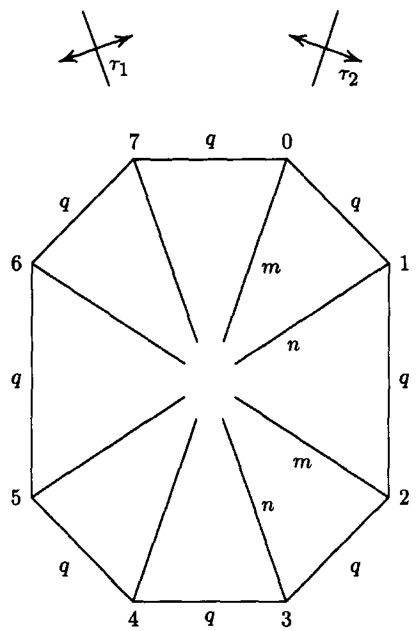
so that $\sigma_0\sigma_3 \in N$. But

$$\sigma_0\sigma_3 = (\tau_0\sigma_3)^2 = (\varrho_1\varrho_0)^2 \in \langle\varrho_0, \varrho_1, \varrho_2\rangle,$$

so that taking the quotient by N leads also to a collapse of the facets of $\mathcal{L}$. This proves that $\mathcal{P}$ cannot exist. Note that $\langle\alpha, \sigma_0\sigma_3\rangle$ itself is normal in $A(\mathcal{L})$.

From now on let $s,t \geq 3$. The case where s or t is even is settled by (b). Here we can construct an infinite family of regular incidence-polytopes $\mathcal{P}_q (q \geq 2)$ in the class $\langle\{4,4\}_{2m,0},\{4,4\}_{2n,0}\rangle$ from the Coxeter diagram

(12)



by applying the operation

(13) $$\varkappa:(\sigma_0,\ldots,\sigma_7;\tau_1,\tau_2)\rightarrow(\sigma_0,\tau_1,\tau_2,\sigma_3)=:(\varrho_0,\ldots,\varrho_3).$$

The underlying graph of the diagram is an octagon together with its four main diagonals marked m and n, respectively; the centre of the octagon is not a node. The incidence-polytope $\mathcal{P}_q$ is finite only if $q=2$.

The remaining case where s and t are odd seems to be more difficult. We shall see below how the case $s=t$ is related to the construction of the universal $\mathcal{L}_{s,0}=\{\{4,4\}_{s,0},\{4,3\}\}$; see also Section 6 for a generalization of this construction. So far we have not been able to decide the existence for the exceptional case of distinct odd s and t. The results of Section 5 seem to indicate that among these possibly existing exceptional polytopes only $\{\{4,4\}_{3,0},\{4,4\}_{5,0}\}$ and its dual can be finite.

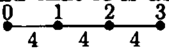
The universal $\mathcal{L}_{s,0}$ was constructed in (3) and (4), with group $A=\langle\varrho_0,\ldots,\varrho_3\rangle$. We apply the operation

(14) $$(\varrho_0,\ldots,\varrho_3)\rightarrow(\varrho_3,\varrho_2\varrho_1\varrho_2,\varrho_0,\varrho_1)=:(\varphi_0,\ldots,\varphi_3)$$

for A; in the terminology of Section 2, this corresponds to applying the halving operation to the vertex-figure of the dual of $\mathcal{L}_{s,0}$. Since s is odd, we can recover the ϱ_i's from the φ_j's; in fact, using (4) we have

(15) $$\begin{cases}\varrho_2&=\sigma_3=\sigma_3(\sigma_1\sigma_0)^s=(\sigma_3\sigma_1\sigma_0)^s=(\sigma_3\tau\sigma_3\tau\sigma_0)^s\\ &=(\varphi_1\varphi_3\varphi_2)^s=(\varphi_3\varphi_1\varphi_2)^s.\end{cases}$$

As a consequence, $\varphi_0, \ldots, \varphi_3$ really generate A and $\langle \varphi_1, \varphi_2, \varphi_3 \rangle$ is the group of the map $\{4,4\}_{s,0}$.

Computing a presentation of A in terms of $\varphi_0, \ldots, \varphi_3$ from a presentation of A in terms of $\varrho_0, \ldots, \varrho_3$ we find that A is abstractly defined by the standard relations for the Coxeter group  and the three extra relations

$$(16) \qquad (\varphi_0\varphi_1\varphi_2\varphi_1)^s = (\varphi_1\varphi_2\varphi_3\varphi_2)^s = 1,$$

$$(17) \qquad (\varphi_0(\varphi_3\varphi_1\varphi_2)^s)^3 = 1.$$

Here we use

$$\begin{aligned} \varphi_1\varphi_0 &= \varrho_2\varrho_1\varrho_2\varrho_3 = \varrho_2(\varrho_1\varrho_2\varrho_3\varrho_2)\varrho_2, \\ \varphi_2\varphi_1\varphi_0\varphi_1 &= \varrho_0\varrho_2\varrho_1(\varrho_2\varrho_3\varrho_2)\varrho_1\varrho_2 \\ &= (\varrho_0\varrho_2\varrho_1\varrho_3)(\varrho_2\varrho_3\varrho_1\varrho_2) = \varrho_2\varrho_3(\varrho_0\varrho_1\varrho_2\varrho_1)\varrho_3\varrho_2 \end{aligned}$$

and

$$\varphi_0(\varphi_3\varphi_1\varphi_2)^s = \varrho_3\varrho_2.$$

The proof of the intersection property for A and its generators $\varphi_0, \ldots, \varphi_3$ is straightforward but very tedious, so that we omit it here; it uses the corresponding property for the underlying Coxeter group (3).

Our considerations imply that A is the group of a regular 4-incidence-polytope $\mathcal{L}$ in $\langle \{4,4\}_{s,0}, \{4,4\}_{s,0} \rangle$. The construction shows that it can be thought of as some kind of skew polytope related to $\mathcal{L}_{s,0}$. The incidence-polytope $\mathcal{L}$ is finite only if $s = 3$, with group $S_6 \times C_2$. In particular this shows that the universal $\{\{4,4\}_{s,0}, \{4,4\}_{s,0}\}$ exists for all odd s and is infinite if $s > 3$. In [3] the universal $\{\{4,4\}_{3,0}, \{4,4\}_{3,0}\}$ was proved to be finite, with a group of order 1440. Comparison with our results shows that $\mathcal{L} = \{\{4,4\}_{3,0}, \{4,4\}_{3,0}\}$, so that (17) can be omitted from the presentation. For $s > 3$ the incidence-polytope $\mathcal{L}$ seems to be distinct from the universal. This is indicated by the existence of 3-cycles in the graph (of all vertices and edges) of $\mathcal{L}$. If $\alpha := (\varphi_3\varphi_1\varphi_2)^s$ and F_0 and F_1 are the vertex and edge in the base flag of $\mathcal{L}$, then, by (17), $\alpha\varphi_0\alpha = \varphi_0\alpha\varphi_0$ stabilizes the vertex $\varphi_0(F_0)$ of F_1 and interchanges the vertices F_0 and $\alpha\varphi_0(F_0)$ of $\alpha(F_1)$, and φ_0 stabilizes $\alpha\varphi_0(F_0)$ while interchanging the vertices F_0 and $\varphi_0(F_0)$ of F_1; hence F_0, $\varphi_0(F_0)$, $\alpha\varphi_0(F_0)$ give a 3-cycle with edges F_1, $\alpha(F_1)$ and $\varphi_0\alpha(F_1)$.

Theorem 3. *(a) For odd $t \geq 3$ the universal $\{\{4,4\}_{2,0}, \{4,4\}_{t,0}\}$ does not exist.*
(b) The universal $\{\{4,4\}_{s,0}, \{4,4\}_{s,0}\}$ exists for all odd $s \geq 3$. The only finite instance is $\{\{4,4\}_{3,0}, \{4,4\}_{3,0}\}$, with group $S_6 \times C_2$.

5. Regular tessellations in polytopes

In this section we shall associate with the polytopes $\{\{4,4\}_{s,t}, \{4,3\}\}$ and $\{\{4,4\}_{q,r}, \{4,4\}_{s,t}\}$ a regular tessellation $\mathcal{H}$ on the 2-sphere or in the euclidean or hyperbolic plane. In a sense which we make precise below, $\mathcal{H}$ cuts right through the polytope. It is remarkable that the polytopes become finite if and only if $\mathcal{H}$ is spherical. It is worth pointing out that our considerations below will not imply the existence of the polytopes; in this respect we need to refer to earlier sections. We begin by discussing the type $\{4,4,4\}$.

5.1. The type $\{4,4,4\}$

Let $\mathcal{P}$ be a regular 4-incidence-polytope of type $\{p,q,r\}$, whose facets have 2-chains of lengths k and whose vertex-figures have Petrie-polygons of lengths m. Again, let its group be $A(\mathcal{P}) = \langle \varrho_0, \dots, \varrho_3 \rangle$. Then the operation

$$(18) \qquad (\varrho_0, \dots, \varrho_3) \to (\varrho_0, \varrho_1\varrho_2\varrho_1, \varrho_3\varrho_2\varrho_3) := (\psi_0, \psi_1, \psi_2)$$

gives a subgroup $\langle \psi_0, \psi_1, \psi_2 \rangle$ of $A(\mathcal{P})$ which is the group of a regular map $\mathcal{H} = \mathcal{H}(\mathcal{P})$ of type $\{k,m\}$ or $\{k, m/2\}$ if m is odd or even, respectively. In fact,

$$\psi_0\psi_1 = \varrho_0\varrho_1\varrho_2\varrho_1$$

and

$$\psi_1\psi_2 = \varrho_1\varrho_2\varrho_1\varrho_3\varrho_2\varrho_3 = (\varrho_1\varrho_2\varrho_3)^2.$$

Let $F_0, \dots, F_3$ be the (proper) faces in the base flag of $\mathcal{P}$. The map $\mathcal{H}$ can be constructed from $\mathcal{P}$ by Wythoff's construction with initial vertex F_0 (cf. [7], [14]). Then the base flag of $\mathcal{H}$ is given by $G_0 = F_0$, $G_1 = F_1$ and the 2-chain

$$G_2 = \{(\varrho_0\varrho_1\varrho_2\varrho_1)^j(F_0) \mid j = 0, \dots, k-1\}$$

of the 3-face F_3 of $\mathcal{P}$. The map itself is given by the transforms of the G_i's by the group $A(\mathcal{H})$. The neighbouring vertices of F_0 in $\mathcal{H}$ are

$$(\psi_1\psi_2)^j(\psi_0(F_0)) = (\varrho_1\varrho_2\varrho_3)^{2j}(\varrho_0(F_0)),$$

for $j = 1, \dots, m$ or $j = 1, \dots, m/2$ if m is odd or even, respectively; that is, as we go around F_0 in $\mathcal{H}$ we pick any other vertex from a Petrie-polygon of the vertex-figure of $\mathcal{P}$ at F_0, eventually all the vertices of the Petrie-polygon if m is odd. If we span topological discs into the 2-faces of $\mathcal{H}$, we can think of $\mathcal{H}$ as a surface which (in a sense) cuts right through $\mathcal{P}$.

If $\mathcal{P}$ is of type $\{p,4,4\}$, then ϱ_2 commutes with ψ_0, ψ_1 and ψ_2, so that $\mathcal{H}$ is invariant under ϱ_2. Hence, in some sense we can think of $\mathcal{H}$ as lying on the reflexion wall of ϱ_2.

Now let $\mathcal{P} = \{\{4,4\}_{s,0}, \{4,4\}_{t,t}\}$. Then $k = s$ and $m = 2t$, so that $\mathcal{H}(\mathcal{P})$ is of type $\{s,t\}$. We shall show that $\mathcal{H}(\mathcal{P})$ is the tessellation $\{s,t\}$ on the 2-sphere or the euclidean or hyperbolic plane. In fact, in the notation of (8) and (9) we have

$$(19) \qquad (\psi_0, \psi_1, \psi_2) = (\sigma_3, \tau_0\sigma_1\tau_0, \tau_1\sigma_1\tau_1) = (\sigma_3, \sigma_4, \sigma_5).$$

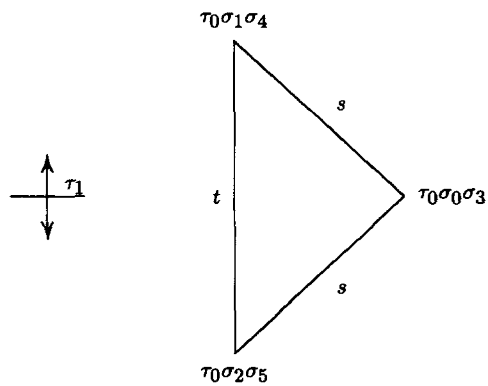
But these are precisely the generators of the Coxeter group which defines the tesselation $\{s,t\}$.

Similarly we can find the map $\mathcal{H}(\mathcal{P}^*)$ which is related to the dual $\mathcal{P}^* = \{\{4,4\}_{t,t}, \{4,4\}_{s,0}\}$ of $\mathcal{P}$. Then, by (8) and (9) the generators of its group are given by

$$(20) \qquad (\psi_0, \psi_1, \psi_2) = (\tau_1, \sigma_1\tau_0\sigma_1, \sigma_3\tau_0\sigma_3) = (\tau_1, \tau_0\sigma_4\sigma_1, \tau_0\sigma_0\sigma_3).$$

To see that these generators define the regular tessellation $\{2s, t\}$ observe that τ_1 acts as an automorphism on the subgroup U of $\langle \psi_0, \psi_1, \psi_2 \rangle$ with diagram

(21)



The group U is really the Coxeter group with diagram (21), since this is true for the group $\langle \sigma_1\sigma_4, \sigma_2\sigma_5, \sigma_0\sigma_3 \rangle$. But adjoining τ_1 to this Coxeter group gives the group of $\{2t, s\}$, so that $\langle \psi_0, \psi_1, \psi_2 \rangle$ must coincide with this group.

For $\mathcal{P} = \{\{4,4\}_{s,s}, \{4,4\}_{t,t}\}$ there is a homomorphism of its group onto the group of $\{\{4,4\}_{s,s}, \{4,4\}_{t,0}\}$ which maps corresponding generators onto each other. Hence, the two corresponding maps $\mathcal{H}$ are similarly related. Since the map for $\mathcal{P}$ is again of type $\{2s, t\}$, it must also be the tessellation $\{2s, t\}$.

Now, let $\mathcal{P} = \{\{4,4\}_{s,0}, \{4,4\}_{t,0}\}$, with $s = 2n$ even and t arbitrary. Since the map $\{4,4\}_{n,n}$ has 2-chains of length $2n$, there is a homomorphism of $A(\mathcal{P})$ onto the group of $\mathcal{L} = \{\{4,4\}_{n,n}, \{4,4\}_{t,0}\}$. This in turn induces a homomorphism of $A(\mathcal{H}(\mathcal{P}))$ onto $A(\mathcal{H}(\mathcal{L}))$. Since $\mathcal{H}(\mathcal{L})$ is a tessellation $\{2n, t\} = \{s, t\}$ and $\mathcal{H}(\mathcal{P})$ is of the same type, $\mathcal{H}(\mathcal{P})$ must coincide with $\{s, t\}$.

Finally, let $\mathcal{P} = \{\{4,4\}_{s,0}, \{4,4\}_{s,0}\}$, with s odd. Let $\mathcal{L}$ denote the regular incidence-polytope in $\langle \{4,4\}_{s,0}, \{4,4\}_{s,0} \rangle$ whose group $A(\mathcal{L}) = \langle \varphi_0, \ldots, \varphi_3 \rangle$ was constructed in (14). Then, in the notation of (14), (3) and (4), the generators of the group of $\mathcal{H}(\mathcal{L})$ are given by

$$(\psi_0, \psi_1, \psi_2) = (\varphi_0, \varphi_1\varphi_2\varphi_1, \varphi_3\varphi_2\varphi_3) = (\sigma_2, \sigma_3\sigma_4\sigma_3, \sigma_4).$$

Since s is odd, $\sigma_3\sigma_4\sigma_3$ and σ_4 generate $\langle \sigma_3, \sigma_4 \rangle$, so that

$$\langle \psi_0, \psi_1, \psi_2 \rangle = \langle \sigma_2, \sigma_3, \sigma_4 \rangle.$$

Then it follows immediately that $\langle \psi_0, \psi_1, \psi_2 \rangle$ is the group of a tessellation $\{s, s\}$; one way to see this is to apply Wythoff's construction suitably to the tessellation $\{3, s\}$ which is defined by $\langle \sigma_2, \sigma_3, \sigma_4 \rangle$. Hence, $\mathcal{H}(\mathcal{L}) = \{s, s\}$. Since $\mathcal{H}(\mathcal{P})$ is of the same type $\{s, s\}$, we also have $\mathcal{H}(\mathcal{P}) = \{s, s\}$.

Our next theorem gives a simple criterion for the finiteness of the known universal incidence-polytopes of type $\{4, 4, 4\}$. So far we proved that

$$(22) \qquad \mathcal{H}(\mathcal{P}) = \begin{cases} \{2s, t\} & \text{if } \mathcal{P} = \{\{4,4\}_{s,s}, \{4,4\}_{t,0}\} \text{ or} \\ & \quad \{\{4,4\}_{s,s}, \{4,4\}_{t,t}\},\ s, t \geq 2; \\ \{s, t\} & \text{if } \mathcal{P} = \{\{4,4\}_{s,0}, \{4,4\}_{t,0}\},\ s \text{ even or } s = t \text{ odd.} \end{cases}$$

Clearly, if $\mathcal{P}$ is finite, then $\mathcal{H}(\mathcal{P})$ is spherical; this proves necessity of the theorem. The sufficiency follows from the classification in the earlier sections. Recall for (c) that $\{\{4,4\}_{2,0},\{4,4\}_{t,0}\}$ does not exist if t is odd.

Theorem 4. *The following is a necessary and sufficient condition for the universal* $\mathcal{P}$ *to be finite.*

(a) $1/2s + 1/t > 1/2$ *if* $\mathcal{P} = \{\{4,4\}_{s,s},\{4,4\}_{t,0}\}$;

(b) $1/2s + 1/t > 1/2$ *and* $1/s + 1/2t > 1/2$ *if* $\mathcal{P} = \{\{4,4\}_{s,s},\{4,4\}_{t,t}\}$;

(c) $1/s + 1/t > 1/2$ *if* $\mathcal{P} = \{\{4,4\}_{s,0},\{4,4\}_{t,0}\}$ *with* s *even or* $s = t$ *odd.*

Note that for (a) the map for the dual $\{\{4,4\}_{t,0},\{4,4\}_{s,s}\}$ is not $\{t,2s\}$ but $\{t,s\}$. Hence, for the infinite $\mathcal{P} = \{\{4,4\}_{3,3},\{4,4\}_{3,0}\}$ the map $\mathcal{H}(\mathcal{P}) = \{6,3\}$ is infinite while $\mathcal{H}(\mathcal{P}^*) = \{3,3\}$ is still finite.

Let us comment on the exceptional type $\mathcal{P}_{s,t} := \{\{4,4\}_{s,0},\{4,4\}_{t,0}\}$ with s,t odd and distinct. If we assume existence of $\mathcal{P}_{s,t}$, it seems likely that the associated map $\mathcal{H}$ is the tessellation $\{s,t\}$. Hence, $\mathcal{P}_{s,t}$ could be finite only if $(s,t) = (3,5)$ or $(5,3)$. Our above example shows that the map $\mathcal{H}$ can be finite even if the polytope is not. However, we believe that the following conjecture is true.

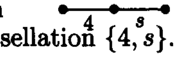
Conjecture: For odd distinct s and t the universal $\mathcal{P}_{s,t}$ exists. The only finite instances are $\mathcal{P}_{3,5}$ and its dual $\mathcal{P}_{5,3}$ (with probably a large group).

5.2. The type $\{4,4,3\}$

Similarly we can proceed with the regular incidence-polytopes $\mathcal{L}_{s,t} = \{\{4,4\}_{s,t},\{4,3\}\}$ and their duals.

For the dual $\mathcal{L}^*_{s,0}$ of $\mathcal{L}_{s,0}$ we can use the construction in (3) and (4) to find the generators for $A(\mathcal{H}(\mathcal{L}^*_{s,0}))$,

$$(\psi_0,\psi_1,\psi_2) = (\sigma_2,\sigma_3\tau\sigma_3,\sigma_0\tau\sigma_0) = (\sigma_2,\sigma_1\sigma_3\tau,\tau\sigma_0\sigma_4).$$

Since $\langle\sigma_2,\sigma_1\sigma_3,\sigma_0\sigma_4\rangle$ is a Coxeter group with diagram  , this must also be true for $\langle\psi_0,\psi_1,\psi_2\rangle$. Hence, $\mathcal{H}(\mathcal{L}^*_{s,0})$ is the tessellation $\{4,s\}$. Note that for $\mathcal{L}_{s,0}$ itself the map $\mathcal{H}$ is of type $\{s,3\}$.

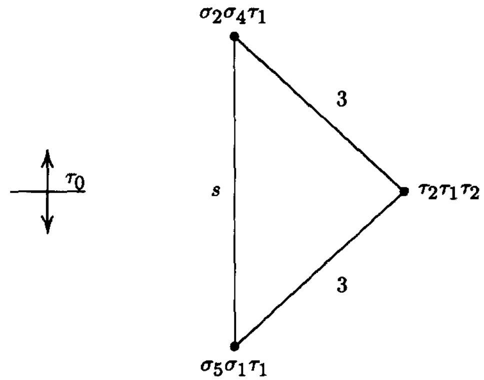
By similar arguments to those in Section 5.1, $A(\mathcal{L}^*_{s,0})$ is a quotient of $A(\mathcal{L}^*_{s,s})$, and so is $A(\mathcal{H}(\mathcal{L}^*_{s,0}))$ or $A(\mathcal{H}(\mathcal{L}^*_{s,s}))$. Since $\mathcal{H}(\mathcal{L}^*_{s,s})$ is of type $\{4,s\}$, this proves that $\mathcal{H}(\mathcal{L}^*_{s,s}) = \{4,s\}$.

For $\mathcal{L}_{s,s}$ itself the map is of type $\{2s,3\}$. We can use (5) and (6) to find the generators,

$$(\psi_0,\psi_1,\psi_2) = (\tau_o,\sigma_2\tau_1\sigma_2,\tau_2\tau_1\tau_2) = (\tau_0,\sigma_2\sigma_4\tau_1,\tau_2\tau_1\tau_2).$$

To see that $\mathcal{H}(\mathcal{L}_{s,s}) = \{2s, 3\}$ observe that in $\langle\psi_0, \psi_1, \psi_2\rangle$ the generator $\psi_0 = \tau_0$ acts as an automorphism on the subgroup illustrated by the diagram

(23)



This subgroup is really a Coxeter group; this can be checked by studying the "rotation subgroup" generated by the pairwise prodncts of the generators.

Theorem 5. *The following is a necessary and sufficient condition for the universal* $\mathcal{L}_{s,t} = \{\{4,4\}_{s,t}, \{4,3\}\}$ *to be finite.*
(a) $1/4 + 1/s > 1/2$ *for* $\mathcal{L}_{s,0}$*;*
(b) $1/2s + 1/3 > 1/2$ *for* $\mathcal{L}_{s,s}$.

The necessity of the conditions follows from the above considerations. The sufficiency uses the classification in Section 3.

Note that besides (18) there are other ways of relating regular maps to the polytopes. While (18) uses the "reflexion hyperplane" of ϱ_2 to define the maps (at least for the types $\{p, 4, 4\}$), the following operation relates in a similar way to ϱ_3:

(24) $$(\varrho_0, \dots, \varrho_3) \to (\varrho_0, \varrho_1, \varrho_2\varrho_3\varrho_2) =: (\alpha_0, \alpha_1, \alpha_2).$$

For example, for $\mathcal{L}^*_{s,t}$ this gives the tessellation $\{3, s\}$ for $t = 0$ or $\{3, 2s\}$ for $s = t$.

6. Relations among the polytopes of type $\{3, 4, 4\}$ and $\{4, 4, 4\}$

In this section we investigate relations among the regular polytopes of type $\{3, 4, 4\}$ and $\{4, 4, 4\}$. A key role is played by the halving operation η described in Section 2. In particular, we make use of

(24) $$\mathcal{M}^\eta = \begin{cases} \{4,4\}_{n,n} & \text{if } \mathcal{M} = \{4,4\}_{2n,0}, n \geq 2; \\ \{4,4\}_{s,0} = \mathcal{M} & \text{if } \mathcal{M} = \{4,4\}_{s,0}, s \text{ odd}, s \geq 3; \\ \{4,4\}_{s,0} & \text{if } \mathcal{M} = \{4,4\}_{s,s}, s \geq 2 \end{cases}$$

(cf. [15], Section 4.3); here, $\mathcal{M}^\eta$ is the transform of $\mathcal{M}$ under η.

Consider the 3-dimensional hyperbolic tessellations $\{3, 4, 4\}$ and $\{4, 4, 4\}$ (cf. Coxeter [6], p. 199). By simplex dissection, we know that the group $[4, 4, 4]$ of

$\{4,4,4\}$ is a subgroup of index 3 in the subgroup $\{3,4,4\}$ of $\{3,4,4\}$. Indeed, if $[3,4,4] = \langle \alpha_0, \dots, \alpha_3 \rangle$, then $[4,4,4] = \langle \beta_0, \dots, \beta_3 \rangle$ is given by

$$(\alpha_0, \dots, \alpha_3) \to (\alpha_0, \alpha_1\alpha_2\alpha_1, \alpha_3, \alpha_2) =: (\beta_0, \dots, \beta_3).$$

The geometric relationship is as follows. The facets of $\{3,4,4\}$ are octahedra with their vertices on the absolute, and the vertex-figures are horospheric tessellations $\{4,4\}$. The facets of the derived $\{4,4,4\}$ are vertex-figures of $\{3,4,4\}$; note for this that

$$\beta_i = \alpha_1\alpha_0\alpha_{i+1}\alpha_0\alpha_1 \quad (i = 0,1,2).$$

The vertex-figure of $\{4,4,4\}$, on the other hand, is obtained by halving the vertex-figure of $\{3,4,4\}$.

The following theorem generalizes the correspondence of $\{3,4,4\}$ and $\{4,4,4\}$ to regular incidence-polytopes of these types.

Theorem 6. *Let $\mathcal{M} = \{4,4\}_{s,0}$ with $s \geq 4$ even, or $\mathcal{M} = \{4,4\}_{s,s}$ with $s \geq 2$. Let $\mathcal{P}$ be the universal $\{\{3,4\},\mathcal{M}\}$, with group $A(\mathcal{P}) = \langle \varrho_0, \dots, \varrho_3 \rangle$. Consider the subgroups $U := \langle \varphi_0, \dots, \varphi_3 \rangle$, with*

$$\varphi_0 = \varrho_0, \quad \varphi_1 = \varrho_1\varrho_2\varrho_1, \quad \varphi_2 = \varrho_3, \quad \varphi_3 = \varrho_2. \tag{25}$$

Then U is of index 3 in $A(\mathcal{P})$, and is the group of the universal $\{\mathcal{M},\mathcal{M}^\eta\}$. In particular, they are either both finite or both infinite.

Note that we excluded the case of $\{4,4\}_{s,0}$ with s odd; this case was discussed in Section 4.2. By (24), for all other cases we have $\mathcal{M}^\eta \neq \mathcal{M}$. The only finite example covered by the theorem is obtained for $\mathcal{M} = \{4,4\}_{2,2}$; then $\mathcal{M}^\eta = \{4,4\}_{2,0}$.

Proof of the Theorem. By our considerations on $\{3,4,4\}$ and $\{4,4,4\}$, the subgroup U of $A(\mathcal{P})$ has index 1 or 3 in $A(\mathcal{P})$. Modulo the intersection property (2), U gives an incidence-polytope $\mathcal{L}$ in $\langle \mathcal{M},\mathcal{M}^\eta \rangle$; in fact, as above,

$$\varphi_i = \varrho_1\varrho_0\varrho_{i+1}\varrho_0\varrho_1 \quad (i = 0,1,2), \tag{26}$$

so that $\langle \varphi_0, \varphi_1, \varphi_2 \rangle$ is the group of a vertex-figure of $\mathcal{P}$ while $\langle \varphi_1, \varphi_2, \varphi_3 \rangle$ is obtained by halving the vertex-figures of $\mathcal{P}$. Our considerations below show that $\mathcal{L} = \{\mathcal{M},\mathcal{M}^\eta\}$ (in particular, U has property (2)), $|A(\mathcal{P}) : U| = 3$ and $\varrho_1 \notin \langle \varphi_0, \dots, \varphi_3 \rangle$; note that $\{\mathcal{M},\mathcal{M}^\eta\}$ exists, by the results of Section 4.

We next show how to reverse the process. Let $\langle \varphi_0, \dots, \varphi_3 \rangle$ be the group of $\{\mathcal{M},\mathcal{M}^\eta\}$. We now reintroduce ϱ_1. We adjoin an element τ to $\langle \varphi_0, \dots, \varphi_3 \rangle$, with the properties

$$\tau^2 = 1, \quad \tau\varphi_3\tau = \varphi_1, \quad \tau\varphi_2\tau = \varphi_2, \quad (\tau\varphi_0)^3 = 1. \tag{27}$$

For the resulting group A define the generators ϱ_i by

$$(\varrho_0, \dots, \varrho_3) := (\varphi_0, \tau, \varphi_3, \varphi_2).$$

It is easy to check that $\langle \varrho_0, \dots, \varrho_3 \rangle$ is a quotient of the group of $\mathcal{P} = \{\{3,4\},\mathcal{M}\}$, if we observe that

$$\varphi_0\tau\varphi_i\tau\varphi_0 = \varrho_{i+1} \quad (i = 0,1,2). \tag{28}$$

Here we can definitely check that $\langle \varphi_0, \dots, \varphi_3 \rangle$ has index 3 in $\langle \varrho_0, \dots, \varrho_3 \rangle$. Indeed, if we write $\gamma := \varrho_0\varrho_1 = \varphi_0\tau$, we see that

$$\varphi_0\gamma = \gamma^2\varphi_0, \quad \varphi_1\gamma = \gamma\varphi_0\varphi_1\varphi_0, \quad \varphi_2\gamma = \gamma\varphi_2, \quad \varphi_3\gamma = \gamma\varphi_1.$$

It follows that $1, \gamma, \gamma^2$ are the three coset representatives. Note that the same process works equally well for a quotient of the group of $\{\mathcal{M}, \mathcal{M}^\eta\}$, especially for U.

Now, putting the two parts together, we see that $\mathcal{L} = \{\mathcal{M}, \mathcal{M}^\eta\}$ and $|A(\mathcal{P}) : U| = 3$; the element τ corresponds to ϱ_1. Then the theorem follows.

We conclude by considering another connexion between polytopes of type {3,4,4} and $\{4,4,4\}$. Let $\langle \varrho_0, \ldots, \varrho_3 \rangle$ be the group of a polytope $\mathcal{P}$ in $\langle \{4,4\}_{s,0}, \mathcal{M} \rangle$, with $\mathcal{M}$ a map of type $\{4,4\}$. Consider the same operation

$$\varkappa : (\varrho_0, \ldots, \varrho_3) \to (\varrho_0, \varrho_1\varrho_2\varrho_1, \varrho_3, \varrho_2) =: (\varphi_0, \ldots, \varphi_3). \tag{29}$$

Then, modulo the intersection property (2) we end up with a polytope $\mathcal{P}^\varkappa$ of type $\{s, 4, 4\}$ with vertex-figures isomorphic to $\mathcal{M}^\eta$.

The most interesting cases seem to be those with $s = 3$. Here we have

$$\varkappa : \begin{cases} \{\{4,4\}_{3,0}, \{4,4\}_{3,0}\} \xrightarrow{1} \{\{3,4\}, \{4,4\}_{3,0}\}; \\ \{\{4,4\}_{3,0}, \{4,4\}_{2,2}\} \xrightarrow{12} \{\{3,4\}, \{4,4\}_{2,0}\}; \\ \{\{4,4\}_{3,0}, \{4,4\}_{4,0}\} \xrightarrow{48} \{\{3,4\}, \{4,4\}_{2,2}\}. \end{cases} \tag{30}$$

The number attached to the arrow is the index of the new group in the old.

Generally, if $s \geq 3$ and $\mathcal{M} = \{4,4\}_{t,0}$ with t odd, then $\mathcal{M}^\eta = \mathcal{M}$, and the same argument as was used in Section 4.2 in showing that $\{\{3,4\}, \{4,4\}_{3,0}\}$ and $\{\{4,4\}_{3,0}, \{4,4\}_{3,0}\}$ have the same group works here. Namely, since $(\varrho_1\varrho_2)^2 = (\varrho_2\varrho_1)^2$, we have

$$\begin{aligned}(\varphi_3\varphi_1\varphi_2)^t &= (\varrho_2\varrho_1\varrho_2\varrho_1\varrho_3)^t \\ &= [(\varrho_2\varrho_1\varrho_2\varrho_3\varrho_1)(\varrho_1\varrho_2\varrho_1\varrho_2\varrho_3)][(\varrho_2\varrho_1\varrho_2\varrho_3\varrho_1) \cdot \ldots \\ &= (\varrho_2\varrho_1\varrho_2\varrho_3)^t \varrho_1 = \varrho_1.\end{aligned}$$

Hence, $A(\mathcal{P}) = A(\mathcal{P}^\varkappa)$.

Exactly as in the proof of Theorem 6 we can recover a quotient of the group of $\{\{4,4\}_{3,0}, \mathcal{M}\}$ from the group of $\{\{3,4\}, \mathcal{M}^\eta\}$ by adjoining a suitable element τ. Thus one universal group is indeed a subgroup of the other. That completes the proof of (30).

The other finite universal example with $s \geq 3$ is

$$\varkappa : \{\{4,4\}_{4,0}, \{4,4\}_{3,0}\} \xrightarrow{1} \{\{4,4\}_{4,0}, \{4,4\}_{3,0}\}.$$

Here, $\mathcal{P} = \mathcal{P}^\varkappa$.

Finally, if (29) is applied to polytopes $\mathcal{P}$ in $\langle \{4,4\}_{s,0}, \{4,3\} \rangle$ or $\langle \{4,4\}_{s,s}, \{4,3\} \rangle$, then we get types $\{s,3,3\}$ or $\{2s,3,3\}$, respectively. In particular,

$$\varkappa : \begin{cases} \{\{4,4\}_{3,0}, \{4,3\}\} \xrightarrow{12} \{3,3,3\}; \\ \{\{4,4\}_{2,2}, \{4,3\}\} \xrightarrow{4} \{4,3,3\}/2. \end{cases}$$

Here, the first correspondence is obvious. To check the second we use the construction of $\mathcal{L}_{2,2}$ in (5) and (6). In fact, in the notation used there, we have

$$\langle \varphi_0, \ldots, \varphi_3 \rangle = \langle \tau_0, \sigma_2\sigma_4\tau_1, \tau_2, \tau_1 \rangle = \langle \sigma_2\sigma_4, \tau_0, \tau_1, \tau_2 \rangle;$$

the latter is isomorphic to a semi-direct product of C_2^6 by D_6, with each factor C_2 corresponding to one of the 6 short diagonals of a hexagon (equivalent under D_6 to the diagonal with endpoints 2 and 4).

Table 1
(The finite universal regular polytopes of type $\{4,4,3\}$ and $\{4,4,4\}$, *excluding the exceptional case*)
Some entries in the last column are taken from [26].

polytope	number of vertices	number of facets	group order	group
$\{\{4,4\}_{2,0},\{4,3\}\}$	4	6	192	$D_4 \times S_4$
$\{\{4,4\}_{3,0},\{4,3\}\}$	30	20	1440	$S_6 \times C_2$
$\{\{4,4\}_{2,2},\{4,3\}\}$	16	12	768	$C_2 \wr D_6$
$\{\{4,4\}_{2,0},\{4,4\}_{t,t}\}$ $(t \geq 2)$	4	$2t^2$	$64t^2$	$(D_t \times D_t \times C_2 \times C_2) \ltimes (C_2 \times C_2)$
$\{\{4,4\}_{2,0},\{4,4\}_{2s,0}\}$ $(s \geq 1)$	4	$4s^2$	$128s^2$	$C_2^2 \ltimes [4,4]_{2,0}$ if $s=1$; $(D_s \times D_s) \ltimes [4,4]_{2,0}$ if $s \geq 2$
$\{\{4,4\}_{3,0},\{4,4\}_{3,0}\}$	20	20	1440	$S_6 \times C_2$
$\{\{4,4\}_{3,0},\{4,4\}_{4,0}\}$	288	512	36864	$C_2 \wr [4,4]_{3,0}$
$\{\{4,4\}_{3,0},\{4,4\}_{2,2}\}$	36	32	2304	$(S_4 \times S_4) \ltimes (C_2 \times C_2)$
$\{\{4,4\}_{2,2},\{4,4\}_{2,2}\}$	16	16	1024	$C_2^4 \ltimes [4,4]_{2,2}$
$\{\{4,4\}_{2,2},\{4,4\}_{3,3}\}$	64	144	9216	$C_2^6 \ltimes [4,4]_{3,3}$

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Added in proof:

[26] P. McMULLEN, and E. SCHULTE: Twisted groups and locally toroidal regular polytopes, in preparation.

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